THE RELATIONSHIP BETWEEN D'ATRI AND k-D'ATRI SPACES.

TERESA ARIAS-MARCO AND MARIA J. DRUETTA

ABSTRACT. In this article we continue the study of the geometry of k-D'Atri spaces began by the second author. We generalize some results including those related with properties of Jacobi operators and applications to spaces of Iwasawa type.

The main result we prove is that every k-D'Atri space for some k, $1 \leq k \leq n-1$ is D'Atri. Moreover, it is known that k-D'Atri spaces are related with properties of Jacobi operators as $\mathrm{tr}R_v$, $\mathrm{tr}R_v^2$ be invariant under the geodesic flow. Here we show that $\mathrm{tr}R_v^3$ is also invariant under the geodesic flow. One of the consequences of this fact is that k-D'Atri spaces for some $k \geq 3$ form a proper subclass of D'Atri spaces.

In the case of spaces of Iwasawa type, we show in particular that the condition on M being k-D'Atri for some $k \geq 3$ characterize the symmetric spaces within this class. Thus, there exit no k-D'Atri spaces of Iwasawa type for $k \geq 3$ unless M be symmetric, in this case M is k-D'Atri for all possible $k \geq 1$.

1. Introduction and Preliminaries

Let M be a n-dimensional Riemannian manifold, ∇ the Levi Civita connection and let R denote the associated curvature tensor defined by $R(u,v) = [\nabla_u, \nabla_v] - \nabla_{[u,v]}$ for all $u, v \in TM$. If $|v| = 1$ the Jacobi operators R_v, are defined by $R_v w = R(w,v)v$.

Let m be a fixed point in M and $v \in T_mM$, $|v| = 1$; we denote by γ_v the geodesic in M with $\gamma_v(0) = m$ and $\gamma_v'(0) = v$. Moreover, for each small $t > 0$, we denote by $S_v(t)$ the shape operator (with respect to the outward unit normal field $\gamma_v'(t)$) of the geodesic sphere

$$G_m(t) = \{\gamma_w(t) = \exp_m(tw) : w \in T_mM,\ |w| = 1\}$$

at $\gamma_v(t)$, where $\exp_m$ denotes the geometric exponential map of M.

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D'Atri spaces were introduced by J.E. D'Atri and H.K. Nickerson in [10]. M is called a *D'Atri space* if the local geodesic symmetries (defined as s_m : $\exp_m(tw) \mapsto \exp_m(-tw)$, $t \sim 0$, $w \in T_mM$, $|w| = 1$) preserve the mean curvature of small geodesic spheres centered at m; that is $\mathrm{tr}S_v(t) = \mathrm{tr}S_{-v}(t)$. Obviously, D'Atri spaces are a natural generalization of locally symmetric spaces (where the local geodesic symmetries are isometries) and in dimension two are locally symmetric, so they have constant sectional curvature. See [18] for more references about D'Atri spaces and related topics.

Many characterizations of D'Atri spaces exit but the most relevant for our work was proved by J.E. D'Atri and H.K. Nickerson [10] and was improved by Z.I. Szabó [21]; namely: A Riemannian manifold is a D'Atri space if and only if it satisfies the series of all odd Ledger conditions $L_{2k+1} = 0$, $k \geq 1$. The *Ledger conditions* are an infinite series of curvature conditions derived from the so-called Ledger recurrence formula, which nowadays, have become of a special and important relevance (see [20], [3]). For example, Z.I. Szabó [21] proved that $L_3 = 0$ implies that the manifold is real analytic. Moreover, the first author and O. Kowalski [4] classified the 4-dimensional homogeneous Riemannian manifolds which satisfy $L_3 = 0$ and used the result to classify the 4-dimensional homogeneous D'Atri spaces, as well (see also [1], [2]).

In section 2 we study properties of the operator $C_v(r) = rS_v(r)$, $r > 0$. In particular, those related with Ledger's conditions which play an important role to prove our main results Theorem 3.1 and Theorem 3.2, developed in Section 3.

M is called a D'Atri space of type k or a k-D'Atri space, $1 \leq k \leq n-1$, if the geodesic symmetries preserve the k-th elementary symmetric functions of the eigenvalues of the shape operators of all small geodesic spheres. Recall, that the k-th elementary symmetric function σ_k, $k = 1, ..., n$, of the eigenvalues of a symmetric endomorphism A on a n-dimensional real vectorial space are determined by its characteristic polynomial as follows,

$$\det(\lambda I - A) = \lambda^n - \sigma_1(A)\lambda^{n-1} + ... + (-1)^k\sigma_k(A)\lambda^{n-k} + ... + (-1)^n\sigma_n(A),$$

$$\sigma_k(A) = \sum_{i_1<i_2<....<i_k} \lambda_{i_1}(A)\cdots\lambda_{i_k}(A)$$

with $1 \leq i_1 < i_2 < ... < i_k \leq n$ and $\lambda_1(A), ..., \lambda_n(A)$ the n eigenvalues of A. Thus, M is a k-D'Atri space if

$$\sigma_k(S_v(r)) = \sigma_k(S_{-v}(r)) \text{ for all unit vector } v \in T_mM$$

where $S_{\pm v}(r)$ denotes the shape operators of $G_m(r)$ at the points $\exp_m r(\pm v)$. Therefore, the 1-D'Atri property is obviously the D'Atri condition.

D'Atri space of type k definitions were introduced by O. Kowalski, F. Prüfer and L. Vanhecke in [18] as a natural analogues of the concept of D'Atri space and it was started as open problem to check if all these analog notions are equivalent or not. The first attemp to solve this open problem have been done by the second author in [14] where it is proved that the notions

of D'Atri spaces (1-D'Atri) and 2-D'Atri spaces are equivalent. Now, we continue such study giving the relation between D'Atri and k-D'Atri spaces for each $k \geq 3$. In Section 3 the main result we prove (Theorem 3.1) is that k-D'Atri spaces (for some $k \geq 2$) are D'Atri. Moreover, k-D'Atri spaces are related with properties of Jacobi operators as $\mathrm{tr}R_v$, $\mathrm{tr}R_v^2$ be invariant under the geodesic flow. Now, we obtain (Theorem 3.2) the same result for $\mathrm{tr}R_v^3$. Note that throughout the paper we can assume $n \geq 3$ and $k \geq 2$.

One of the consequences of Theorem 3.2 is that k-D'Atri spaces for some $k \geq 3$ form a proper subclass of D'Atri spaces. To prove it, we work on spaces of Iwasawa type in Section 4 where we characterize k-D'Atri spaces for $k \geq 3$ as the symmetric spaces.

Some properties of k-D'Atri spaces ($k \geq 1$) under the Iwasawa type hypothesis have been study in [13] and [14]. In Section 4, we continue such study proving that every D'Atri space where $\mathrm{tr}R_v^3$ is invariant under the geodesic flow is a symmetric space. Moreover, we also get that $\mathfrak{C}$-spaces of Iwasawa type which are also D'Atri spaces are symmetric spaces.

$\mathfrak{C}$-spaces were introduced by J. Berndt and L.Vanhecke in [6]. By definition, M is a $\mathfrak{C}$-*space* if the eigenvalues of $R_{\gamma_v'(t)}$ are constant along each geodesic. For locally symmetric spaces this is always the case, so $\mathfrak{C}$-spaces are another natural generalization of locally symmetric spaces. On the other hand, in the case of rank one it was shown in [11] that $\mathfrak{C}$-spaces of Iwasawa type are symmetric. Moreover, Damek-Ricci spaces are spaces of Iwasawa type and algebraic rank 1 and the non-symmetric ones were the first examples of D'Atri spaces which are not $\mathfrak{C}$-spaces [5]. However, it is an open question whether a $\mathfrak{C}$-space is a D'Atri space.

2. Properties of $C_v(r) = rS_v(r)$ and Ledger's conditions

Let $v \in T_mM$ be a unit vector and consider a small real $r > 0$. If M is real analytic, then it is well known that the endomorphism $C_v(r) = rS_v(r) = \sum_{k=0}^{\infty} \alpha_k(v) r^k$ with $\alpha_k(v) = \frac{1}{k!}C_v^{(k)}(0)$, gives the power series expansion of $C_v(r)$ at $r = 0$, where

$$C_v^{(k)}(0) = \frac{D^k}{dr^k}\, C_v(r)\big|_{r=0}\,, \quad k \geq 1$$

may be computed by using the recursion formula of Ledger that is given by

$$(1)\quad (k+1)C_v^{(k)}(0) = -k(k-1)R_v^{(k-2)} - \sum_{l=2}^{k-2} \tbinom{k}{l} C_v^{(l)}(0)C_v^{(k-l)}(0) \text{ for } k \geq 2,$$

with $C_v(0) = \mathrm{Id}$, $C_v'(0) = 0$ (see [7, 9, 18]). Here we use the notation $R_v = R_v(0)$ and $R_v^{(k)} = R_v^{(k)}(0)$, $k \geq 1$, the k-th covariant derivative of the tensor $R_{\gamma_v'(t)}$ along γ_v at $t = 0$. Then,

$$\begin{aligned} R_v' &= R_v'(0) = \left(\nabla_{\gamma_v'(t)} R_{\gamma_v'(t)}\right)\Big|_{t=0} \quad \text{and} \\ R_v^{(k)} &= R_v^{(k)}(0) = \left(\nabla_{\gamma_v'(t)} R^{(k-1)}_{\gamma_v'(t)}\right)\Big|_{t=0} \text{ for all } k \geq 2. \end{aligned}$$

Thus, from formula (1) we have

$$
\begin{aligned}
&C_v(0) = \mathrm{Id}, \quad C'_v(0) = 0, \quad C''_v(0) = \tfrac{-2}{3} R_v, \quad C_v^{(3)}(0) = -\tfrac{3}{2} R'_v, \\
&C_v^{(4)}(0) = \tfrac{-4}{5}\left(3R''_v + \tfrac{2}{3} R_v \circ R_v\right), \\
&C_v^{(5)}(0) = \tfrac{-5}{3}\left(2R_v^{(3)} + R'_v \circ R_v + R_v \circ R'_v\right), \\
&C_v^{(6)}(0) = \tfrac{-3}{7}\left(10R_v^{(4)} + 8R_v \circ R''_v + 8R''_v \circ R_v + 15R'_v \circ R'_v + \tfrac{32}{9} R_v \circ R_v \circ R_v\right), \\
&C_v^{(7)}(0) = \tfrac{-7}{12}\Big(9R_v^{(5)} + 10R_v \circ R_v^{(3)} + 10R_v^{(3)} \circ R_v + 27R'_v \circ R''_v + 27R''_v \circ R'_v \\
&\qquad\qquad +11R_v \circ R_v \circ R'_v + 11R'_v \circ R_v \circ R_v + 10R_v \circ R'_v \circ R_v\Big).
\end{aligned} \tag{2}
$$

Moreover, the operators $C_v^{(k)}(0)$, $k \geq 1$, satisfy the following identities which will be crucial to prove our main result Theorem 3.1.

Proposition 2.1. *Under the above assumptions, for each unit vector v in T_mM*

$$
\begin{aligned}
C_v^{(2k)}(0) &= C_{-v}^{(2k)}(0) \ \textit{and} \\
C_v^{(2k+1)}(0) &= -\, C_{-v}^{(2k+1)}(0) \ \textit{for } k \geq 1.
\end{aligned} \tag{3}
$$

Equivalently,

$$
\begin{aligned}
\alpha_{2k}(v) &= \alpha_{2k}(v) \ \textit{and} \\
\alpha_{2k+1}(v) &= -\, \alpha_{2k+1}(-v) \ \textit{for } k \geq 1,
\end{aligned}
$$

since $\alpha_l(v) = \frac{1}{l!} C_v^{(l)}(0)$ *for* $l \geq 0$.

Proof. By definition $R_v = R_{-v}$ and $R'_v = -R'_{-v}$. Therefore, from (1) we easily see that $C_v^{(2)}(0) = -\frac{2}{3} R_v = C_{-v}^{(2)}(0)$ and $C_v^{(3)}(0) = -\frac{3}{2} R'_v = -C_{-v}^{(3)}(0)$. Now we will assume that (3) is true for $1 \leq k \leq m-1$ and we will prove it for $k = m$. In the even case, from (1) we get

$$
\begin{aligned}
(2m+1)C_v^{(2m)}(0) &= \\
&= -\,2m(2m-1)R_v^{(2(m-1))} - \sum_{l=2}^{2(m-1)} \binom{2m}{l} C_v^{(l)}(0) C_v^{(2m-l)}(0) \\
&= -\,2m(2m-1)R_v^{(2(m-1))} - \sum_{j=1}^{m-1} \binom{2m}{2j} C_v^{(2j)}(0) C_v^{(2(m-j))}(0) \\
&\quad + \sum_{j=1}^{m-2} \binom{2m}{2j+1} C_v^{(2j+1)}(0) C_v^{(2(m-j)-1)}(0).
\end{aligned} \tag{4}
$$

In the odd case, (1) becomes

$$
\begin{aligned}
&2(m+1)C_v^{(2m+1)}(0) = \\
&\quad = -2m(2m+1)R_v^{(2m-1)} - \sum_{l=2}^{2m-1} \binom{2m+1}{l} C_v^{(l)}(0)C_v^{(2m+1-l)}(0) \\
&\quad = -2m(2m+1)R_v^{(2m-1)} - \sum_{j=1}^{m-1} \binom{2m+1}{2j} C_v^{(2j)}(0)C_v^{(2(m-j)+1)}(0) \\
&\qquad - \sum_{j=1}^{m-1} \binom{2m+1}{2j+1} C_v^{(2j+1)}(0)C_v^{(2(m-j))}(0).
\end{aligned} \tag{5}
$$

Finally, using the induction hypothesis and the facts

$$R_v^{(2k)} = R_{-v}^{(2k)}, \quad R_v^{(2k+1)} = -R_{-v}^{(2k+1)} \quad \text{for } k \geq 1,$$

from (4) and (5) we conclude the thesis. □

On the other hand, using $C_v^{(k)}(0)$, $k \geq 1$, the Ledger conditions can be introduced and the well-known characterization of D'Atri spaces using the odd ones can be proved.

Definition 2.1. *In the previous context if for each unit vector $v \in T_mM$*

$$L_k = \mathrm{tr} C_v^{(k)}(0) = \frac{d^k}{dr^k} \, \mathrm{tr} C_v(r)\big|_{r=0} \,, \; k \geq 1,$$

then $L_{2k+1} = 0$ and $L_{2k} = c_{2k}$, $k \geq 1$, define the Ledger conditions (associated to v) of odd order and even order, respectively, at the point m.

Remark 2.1. *By the previous proposition it is immediate that*

$$C_v(r) - C_{-v}(r) = 2\sum_{k=1}^{\infty} \alpha_{2k+1}(v) r^{2k+1}.$$

Therefore, $0 = \mathrm{tr}(S_v(r)) - \mathrm{tr}(S_{-v}(r)) = \frac{1}{r}\mathrm{tr}(C_v(r) - C_{-v}(r))$, $r > 0$, if and only if $\mathrm{tr}(\alpha_{2k+1}(v)) = 0$ for all $k \geq 1$. That is, M is a D'Atri space if and only if the infinite series of Ledger conditions of odd order are satisfied.

In the remaining of the section, we will show how to use the three first odd Ledger conditions to obtain some useful identities.

Remark 2.2. *If $v \in T_mM$ is a unit vector, then for any geodesic $\gamma_v(t)$,*

$$\frac{d}{dt}\left(\mathrm{tr}(R_{\gamma_v'(t)}^{(k)} \circ R_{\gamma_v'(t)}^{(k)})\right) = 2\mathrm{tr}\left(R_{\gamma_v'(t)}^{(k)} \circ R_{\gamma_v'(t)}^{(k+1)}\right) \text{ for all } k \geq 0.$$

In fact, for any orthonormal parallel basis $\{e_i(t)\}_{i=1}^n$ along $\gamma_v(t)$ with $e_n(t) = \gamma_v'(t)$, using that all operators $R^{(k)}_{\gamma_v'(t)}$ are symmetric, we have

$$\begin{aligned}
\frac{d}{dt}\left(\mathrm{tr}\left(R^{(k)}_{\gamma_v'(t)}\right)^2\right) =& \sum_{i=1}^{n-1}\frac{d}{dt}\left\langle \left(R^{(k)}_{\gamma_v'(t)}\right)^2 e_i(t), e_i(t)\right\rangle \\
=& \sum_{i=1}^{n-1}\frac{d}{dt}\left\langle R^{(k)}_{\gamma_v'(t)} e_i(t), R^{(k)}_{\gamma_v'(t)} e_i(t)\right\rangle \\
=& 2\sum_{i=1}^{n-1}\left\langle \nabla_{\gamma_v'(t)}\left(R^{(k)}_{\gamma_v'(t)} e_i(t)\right), R^{(k)}_{\gamma_v'(t)} e_i(t)\right\rangle \\
=& 2\sum_{i=1}^{n-1}\left\langle \left(\nabla_{\gamma_v'(t)} R^{(k)}_{\gamma_v'(t)}\right) e_i(t), R^{(k)}_{\gamma_v'(t)} e_i(t)\right\rangle \\
=& 2\sum_{i=1}^{n-1}\left\langle R^{(k+1)}_{\gamma_v'(t)} e_i(t), R^{(k)}_{\gamma_v'(t)} e_i(t)\right\rangle = 2\mathrm{tr}\left(R^{(k)}_{\gamma_v'(t)} \circ R^{(k+1)}_{\gamma_v(t)}\right).
\end{aligned}$$

Proposition 2.2. *Let M be a Riemannian manifold. If $L_3 = 0$, then*

(i) $\mathrm{tr}R_v^{(k)} = 0$ *for all* $k \geq 1$. *Consequently,* $\mathrm{tr}R^{(k)}_{\gamma_v'(t)} = 0$ *for all* $k \geq 1$ *along* $\gamma_v(t)$.

(ii) *The second odd Ledger condition $L_5 = 0$ becomes*

$$\mathrm{tr}(R_v \circ R_v') = 0 \text{ for all unit vector } v \in T_mM. \tag{6}$$

Equivalently, $\mathrm{tr}(R_v^2)$ *is invariant under the geodesic (local) flow.*

Proof. The proof of (i) can be seen in [14, Proposition 2.2 (ii)]. To prove (ii), from (2) we see that

$$L_5 = -\tfrac{5}{3}\mathrm{tr}(2R_v^{(3)} + R_v' \circ R_v + R_v \circ R_v') = -\tfrac{10}{3}\left(\mathrm{tr}(R_v^{(3)}) + \mathrm{tr}(R_v \circ R_v')\right).$$

Therefore, if $L_3 = 0$ we obtain by (i) that $L_5 = 0$ and (6) are equivalent.

Finally, due to Remark 2.2, applying (6) for each real $t > 0$ to $\gamma_v'(t)$, the tangent vector to the geodesic $\widetilde{\gamma}_v(s) = \gamma_v(t+s)$, $s \sim 0$, at $s = 0$, we have that

$$\frac{d}{dt}\left(\mathrm{tr}R^2_{\gamma_v'(t)}\right) = 2\mathrm{tr}\left(R_{\gamma_v'(t)} \circ R'_{\gamma_v'(t)}\right) = 0 \quad \text{along} \quad \gamma_v(t),$$

since $|\gamma_v'(t)| = 1$. Thus, $\mathrm{tr}R^2_{\gamma_v'(t)} = \mathrm{tr}R_v^2$ for all possible t, which means that $\mathrm{tr}R_v^2$ is invariant under the geodesic flow wherever the geodesic is defined. □

Remark 2.3. *If $v \in T_mM$ is a unit vector, then for any geodesic $\gamma_v(t)$,*

$$\frac{d}{dt}\left(\mathrm{tr}(R^{(k)}_{\gamma_v'(t)} \circ R_{\gamma_v'(t)})\right) = \mathrm{tr}\left(R^{(k+1)}_{\gamma_v'(t)} \circ R_{\gamma_v'(t)} + R^{(k)}_{\gamma_v'(t)} \circ R'_{\gamma_v'(t)}\right), \ k \geq 1.$$

In fact, for any orthonormal parallel basis $\{e_i(t)\}_{i=1}^n$ *along* $\gamma_v(t)$ *with* $e_n(t) = \gamma_v'(t)$*, we compute*

$$
\begin{aligned}
&\frac{d}{dt}\left(\operatorname{tr}\left(R^{(k)}_{\gamma_v'(t)} \circ R_{\gamma_v'(t)}\right)\right) = \\
&= \sum_{i=1}^{n-1} \frac{d}{dt}\left\langle \left(R^{(k)}_{\gamma_v'(t)} \circ R_{\gamma_v'(t)}\right) e_i(t), e_i(t)\right\rangle \\
&= \sum_{i=1}^{n-1} \frac{d}{dt}\left\langle R_{\gamma_v'(t)} e_i(t), R^{(k)}_{\gamma_v'(t)} e_i(t)\right\rangle \\
&= \sum_{i=1}^{n-1} \left\{\left\langle \nabla_{\gamma_v'(t)}\left(R_{\gamma_v'(t)} e_i(t)\right), R^{(k)}_{\gamma_v'(t)} e_i(t)\right\rangle + \left\langle R_{\gamma_v'(t)} e_i(t), \nabla_{\gamma_v'(t)}\left(R^{(k)}_{\gamma_v'(t)} e_i(t)\right)\right\rangle\right\} \\
&= \sum_{i=1}^{n-1} \left\{\left\langle \left(\nabla_{\gamma_v'(t)} R_{\gamma_v'(t)}\right) e_i(t), R^{(k)}_{\gamma_v'(t)} e_i(t)\right\rangle + \left\langle R_{\gamma_v'(t)} e_i(t), \left(\nabla_{\gamma_v'(t)} R^{(k)}_{\gamma_v'(t)}\right) e_i(t)\right\rangle\right\} \\
&= \sum_{i=1}^{n-1} \left\{\left\langle R'_{\gamma_v'(t)} e_i(t), R^{(k)}_{\gamma_v'(t)} e_i(t)\right\rangle + \left\langle R_{\gamma_v'(t)} e_i(t), R^{(k+1)}_{\gamma_v'(t)} e_i(t)\right\rangle\right\} \\
&= \sum_{i=1}^{n-1} \left\{\left\langle \left(R^{(k)}_{\gamma_v'(t)} \circ R'_{\gamma_v'(t)}\right) e_i(t), e_i(t)\right\rangle + \left\langle \left(R^{(k+1)}_{\gamma_v'(t)} \circ R_{\gamma_v'(t)}\right) e_i(t), e_i(t)\right\rangle\right\} \\
&= \operatorname{tr}\left(R^{(k+1)}_{\gamma_v'(t)} \circ R_{\gamma_v'(t)} + R^{(k)}_{\gamma_v'(t)} \circ R'_{\gamma_v'(t)}\right).
\end{aligned}
$$

Proposition 2.3. *Let* M *be a Riemannian manifold. If* $L_3 = 0$ *and* $L_5 = 0$*, then the third odd Ledger's condition* $L_7 = 0$ *becomes*

$$16\operatorname{tr}(R_v' \circ R_v^2) - 3\operatorname{tr}(R_v' \circ R_v'') = 0 \text{ for all unit vector } v \in T_mM. \tag{7}$$

Consequently, $\operatorname{tr}(32R_v^3 - 9R_v' \circ R_v')$ *is invariant under the geodesic flow.*

Proof. Let $v \in T_mM$ be a unit vector. We first show that if the property $\operatorname{tr}(R_v \circ R_v') = 0$ is fulfilled, then

$$
\begin{aligned}
\operatorname{tr}(R'_{\gamma_v'(t)} \circ R'_{\gamma_v'(t)} + R_{\gamma_v'(t)} \circ R''_{\gamma_v'(t)}) &= 0, \\
\operatorname{tr}(3R''_{\gamma_v'(t)} \circ R'_{\gamma_v'(t)} + R_{\gamma_v'(t)} \circ R^{(3)}_{\gamma_v'(t)}) &= 0
\end{aligned} \tag{8}
$$

along $\gamma_v(t)$. Equivalently,

$$
\begin{aligned}
\operatorname{tr}(R_v' \circ R_v' + R_v \circ R_v'') &= 0, \\
\operatorname{tr}(3R_v'' \circ R_v' + R_v \circ R_v^{(3)}) &= 0.
\end{aligned} \tag{9}
$$

If $\operatorname{tr}(R_v \circ R_v') = 0$ for all unit vector $v \in T_mM$, then by the same argument used in the proof of Proposition 2.2, we have that

$$\operatorname{tr}\left(R_{\gamma_v'(t)} \circ R'_{\gamma_v'(t)}\right) = 0 \quad \text{along} \quad \gamma_v(t),$$

since $|\gamma_v'(t)| = 1$. Then, from Remark 2.3 we get (8) deriving twice the preceding equality. The equivalence is immediate, applying the above argument to the equalities given by (9).

Now, we continue proving (7). From (2) we see that

$$L_7 = -\tfrac{7}{12}\mathrm{tr}(9R_v^{(5)} + 20R_v \circ R_v^{(3)} + 54R_v' \circ R_v'' + 32R_v' \circ R_v^2)$$

and applying (ii) of Proposition 2.2, the condition $L_3 = 0$ gives

$$L_7 = -\tfrac{7}{6}\mathrm{tr}(10R_v \circ R_v^{(3)} + 27R_v' \circ R_v'' + 16R_v' \circ R_v^2).$$

If $L_5 = 0$, it follows from Proposition 2.2 and (9) that L_7 is reduced to

$$\begin{aligned} L_7 &= -\tfrac{7}{6}\mathrm{tr}(-30R_v'' \circ R_v' + 27R_v' \circ R_v'' + 16R_v' \circ R_v^2) \\ &= -\tfrac{7}{6}\mathrm{tr}(-3R_v' \circ R_v'' + 16R_v' \circ R_v^2). \end{aligned}$$

Thus, the condition $L_7 = 0$ and equality (7) are equivalent.

Finally, due to [14, Lemma 2.3] and Remark 2.2, from (7) we get

$$\begin{aligned} &\frac{d}{dt}\left(\mathrm{tr}(32R^3_{\gamma_v'(t)} - 9R'_{\gamma_v'(t)} \circ R'_{\gamma_v'(t)})\right) = \\ &= 6\mathrm{tr}(16R'_{\gamma_v'(t)} \circ R^2_{\gamma_v'(t)} - 3R'_{\gamma_v'(t)} \circ R''_{\gamma_v'(t)}) = 0 \end{aligned}$$

since $|\gamma_v'(t)| = 1$. Thus,

$$\mathrm{tr}(32R^3_{\gamma_v'(t)} - 9R'_{\gamma_v'(t)} \circ R'_{\gamma_v'(t)}) = \mathrm{tr}(32R_v^3 - 9R_v' \circ R_v')$$

along $\gamma_v(t)$, which means that $\mathrm{tr}(32R_v^3 - 9R_v' \circ R_v')$ is invariant under the geodesic flow.

□

3. Some geometric properties of D'Atri spaces of type k

In this section, we will show first that k-D'Atri spaces for some $k \geq 3$ determine a proper subclass of the class of D'Atri spaces despite of D'Atri and 2-D'Atri properties are equivalent (see [14, Proposition 2.1]). We will use the following algebraic fact whose proof can be seen in [14, proof of Proposition 2.2]: Let V be a real n-dimensional vectorial space ($n \geq 2$) and let A be a symmetric operator on V. Then for any real r and any natural $l \geq 1$, the k-th elementary symmetric function of the eigenvalues of the operator $\mathrm{Id} + r^l A$, $1 \leq k \leq n-1$, satisfies

$$\sigma_k(\mathrm{Id} + r^l A) = \tbinom{n}{k} + \tbinom{n-1}{k-1} r^l \mathrm{tr}A + O(r^{2l}). \tag{10}$$

Theorem 3.1. *Let M be a n-dimensional D'Atri space of type k for some $2 \leq k \leq n-1$, then M is a D'Atri space.*

Proof. We will prove that M is a D'Atri space checking by induction that all odd Ledger's conditions are satisfied. Let $v \in T_m M$ be a fix unit vector. By [14, Proposition 2.2] we know that $\mathrm{tr}R_v' = 0$ and consequently, $L_3 = \mathrm{tr}C_v^{(3)}(0) = -\frac{3}{2}\mathrm{tr}R_v' = 0$. Now, we will assume that $L_{2h-1} = \mathrm{tr}C_v^{(2h-1)}(0) = 0$ for $2 \leq h \leq m$ and we will prove that $L_{2(m+1)-1} = \mathrm{tr}C_v^{(2m+1)}(0) = 0$. This

is equivalent to assume that $\mathrm{tr}\alpha_{2h-1}(v) = 0$ for $2 \leq h \leq m$ and to show that $\mathrm{tr}\alpha_{2m+1}(v) = 0$, since $\alpha_k(v) = \frac{1}{k!}C_v^{(k)}(0)$.

We fix a small $t > 0$ and we set $\alpha_j = \alpha_j(v)$, $O(v, t^l) = \sum_{i=l}^{\infty} \alpha_i t^i$ and

$B_t(v) = \frac{1}{t^{m-1}}\alpha_2 + \frac{1}{t^{m-2}}\alpha_3 + \cdots + \frac{1}{t}\alpha_m + \alpha_{m+1} + t\alpha_{m+2} + \cdots + t^{m-1}\alpha_{2m} + t^m\alpha_{2m+1}$.

Now, applying (10) to $B_t(v) + O(v, t^{m+1})$ we obtain

$$\begin{aligned}\sigma_k(C_v(t)) &= \sigma_k(\mathrm{Id} + \alpha_2 t^2 + \alpha_3 t^3 + \cdots + \alpha_{2m+1}t^{2m+1} + O(v, t^{2m+2})) \\ &= \sigma_k\left(\mathrm{Id} + t^{m+1}(B_t(v) + O(v, t^{m+1}))\right) \\ &= \tbinom{n-1}{k} + \tbinom{n-2}{k-1}t^{m+1}\mathrm{tr}\left(B_t(v) + O(v, t^{m+1})\right) + O(v, t^{2m+2}) \\ &= \tbinom{n-1}{k} + \tbinom{n-2}{k-1}t^{m+1}\mathrm{tr}B_t(v) + O(v, t^{2m+2}).\end{aligned}$$

Thus, by induction hypothesis

$$\begin{aligned}\sigma_k(C_v(t)) = \tbinom{n-1}{k} + \tbinom{n-2}{k-1}&\left(t^2\mathrm{tr}\alpha_2(v) + t^4\mathrm{tr}\alpha_4(v) + \cdots +\right. \\ &\left.+t^{2m-2}\alpha_{2m-2}(v) + t^{2m}\alpha_{2m}(v) + t^{2m+1}\alpha_{2m+1}(v)\right) + O(v, t^{2m+2}).\end{aligned}$$

Then, we also have by Proposition 2.1

$$\begin{aligned}\sigma_k(C_{-v}(t)) = \tbinom{n-1}{k} + \tbinom{n-2}{k-1}&\left(t^2\mathrm{tr}\alpha_2(v) + t^4\mathrm{tr}\alpha_4(v) + \cdots +\right. \\ &\left.+t^{2m-2}\alpha_{2m-2}(v) + t^{2m}\alpha_{2m}(v) - t^{2m+1}\alpha_{2m+1}(v)\right) + O(-v, t^{2m+2}).\end{aligned}$$

Therefore, under the assumption that the k-D'Atri property is satisfied and setting $O(\pm v, t) = O(v, t) - O(-v, t)$,

$$\begin{aligned}0 &= \sigma_k(C_v(t)) - \sigma_k(C_{-v}(t)) \\ &= 2\tbinom{n-2}{k-1}t^{2m+1}\mathrm{tr}\alpha_{2m+1}(v) + O(\pm v, t^{2m+2}).\end{aligned}$$

This gives

$$\mathrm{tr}\alpha_{2m+1}(v) + O(\pm v, t) = 0 \text{ for any small } t > 0,$$

which implies that $\mathrm{tr}\alpha_{2m+1}(v) = 0$ or $L_{2(m+1)-1} = \mathrm{tr}C_v^{(2m+1)}(0) = 0$ taking into account that $\lim_{t\to 0} O(\pm v, t) = 0$. □

Remark 3.1. *The converse of the previous theorem does not hold in general. Non-symmetric Damek-Ricci spaces are D'Atri spaces which are not 3-D'Atri (see* [14, Theorem 3.2, (ii)]*). Moreover, in the next section we will prove that non-symmetric Damek-Ricci spaces are not k-D'Atri for any $k \geq 3$ (see Corollary 4.6).*

Now, we will prove a new geometric property of k-D'Atri spaces related to Jacobi operators along geodesics which continues the results of [14, Proposition 2.2], where it is proved that $\mathrm{tr}(R_v)$, $\mathrm{tr}(R_v^2)$ are invariant under the geodesic flow. We note that by [17] and [6], D'Atri spaces in dimension 3 are homogeneous and have the property that the eigenvalues of the Jacobi operator are constant along each geodesic (i.e.; they are $\mathfrak{C}$-spaces). Thus, $\mathrm{tr}(R_v^3)$ is invariant under the geodesic flow and the next theorem is also valid

for $n=3$. We will use Newton's relations (see [8, A.IV.70]): For n real numbers, $\lambda_1,...,\lambda_n$, and any natural $k=1,...,n$, if we denote by $s_k=\sum_{i=1}^n \lambda_i^k$ and by σ_k their associated k-th elementary symmetric functions, then

$$s_k - s_{k-1}\sigma_1 + s_{k-2}\sigma_2 + ... + (-1)^{k-1}s_1\sigma_{k-1} + (-1)^k k\sigma_k = 0, \;\; k \le n. \tag{11}$$

Theorem 3.2. *If M is a n-dimensional D'Atri space of type k with $n \ge 4$ and $3 \le k \le n-1$, then*

$$\mathrm{tr}(R_v' \circ R_v^2) = 0 \text{ for all unit vector } v \in T_mM. \tag{12}$$

Equivalently, $\mathrm{tr}(R_v^3)$ *is invariant under the geodesic flow.*

Proof. Let $v \in T_mM$ be a fix unit vector and let $t>0$ be a fix small real number. We set $\alpha_j=\alpha_j(v)$, $O(v,t^l)=\sum_{i=l}^\infty \alpha_i t^i$ and we expand $C_v(t)^l$ for $l \ge 1$,

$$\begin{aligned} C_v(t)^l &= \left(I+\alpha_2t^2+\alpha_3t^3+\alpha_4t^4+\alpha_5t^5+\alpha_6t^6+\alpha_7t^7+O(v,t^8)\right)^l \\ &= I + \binom{l}{1}\{\alpha_2t^2+\alpha_3t^3+\alpha_4t^4+\alpha_5t^5+\alpha_6t^6+\alpha_7t^7+O(v,t^8)\} \\ &\quad + \binom{l}{2}\{\alpha_2t^2+\alpha_3t^3+\alpha_4t^4+\alpha_5t^5+\alpha_6t^6+\alpha_7t^7+O(v,t^8)\}^2 \\ &\quad + \binom{l}{3}\{\alpha_2t^2+\alpha_3t^3+\alpha_4t^4+\alpha_5t^5+\alpha_6t^6+\alpha_7t^7+O(v,t^8)\}^3 \\ &\quad + \cdots \end{aligned}$$

That is,

$$\begin{aligned} C_v(t)^l = \; & I + t^2\binom{l}{1}\alpha_2 + t^3\binom{l}{1}\alpha_3 + t^4\left\{\binom{l}{1}\alpha_4 + \binom{l}{2}\alpha_2^2\right\} \\ & + t^5\left\{\binom{l}{1}\alpha_5 + \binom{l}{2}(\alpha_2\alpha_3+\alpha_3\alpha_2)\right\} \\ & + t^6\left\{\binom{l}{1}\alpha_6 + \binom{l}{2}(\alpha_2\alpha_4+\alpha_4\alpha_2+\alpha_3^2) + \binom{l}{3}\alpha_2^3\right\} \\ & + t^7\left\{\binom{l}{1}\alpha_7 + \binom{l}{2}(\alpha_2\alpha_5+\alpha_5\alpha_2+\alpha_3\alpha_4+\alpha_4\alpha_3)\right. \\ & \quad \left. + \binom{l}{3}(\alpha_2^2\alpha_3+\alpha_2\alpha_3\alpha_2+\alpha_3\alpha_2^2)\right\} + O(v,t^8) \end{aligned}$$

Hence, setting $s_l=s_l(v)$ and $\gamma_j=\gamma_j(v)$, we have

$$
\begin{aligned}
s_l = \operatorname{tr} C_v(t)^l = \; & n - 1 + t^2 \binom{l}{1}\gamma_1 + t^3 \binom{l}{1}\gamma_2 + t^4 \left\{ \binom{l}{1}\gamma_3 + \binom{l}{2}\gamma_4 \right\} \\
& + t^5 \left\{ \binom{l}{1}\gamma_5 + \binom{l}{2}\gamma_6 \right\} \\
& + t^6 \left\{ \binom{l}{1}\gamma_7 + \binom{l}{2}\gamma_8 + \binom{l}{3}\gamma_9 \right\} \\
& + t^7 \left\{ \binom{l}{1}\gamma_{10} + \binom{l}{2}\gamma_{11} + \binom{l}{3}\gamma_{12} \right\} + O(v, t^8)
\end{aligned}
$$

where
(13)

$$
\begin{gathered}
\gamma_1 = \operatorname{tr}\alpha_2, \quad \gamma_2 = \operatorname{tr}\alpha_3, \quad \gamma_3 = \operatorname{tr}\alpha_4, \quad \gamma_4 = \operatorname{tr}(\alpha_2^2), \quad \gamma_5 = \operatorname{tr}\alpha_5, \\
\gamma_6 = 2\operatorname{tr}(\alpha_2\alpha_3), \quad \gamma_7 = \operatorname{tr}\alpha_6, \quad \gamma_8 = 2\operatorname{tr}(\alpha_2\alpha_4) + \operatorname{tr}(\alpha_3^2), \quad \gamma_9 = \operatorname{tr}(\alpha_2^3), \\
\gamma_{10} = \operatorname{tr}\alpha_7, \quad \gamma_{11} = 2\operatorname{tr}(\alpha_2\alpha_5) + 2\operatorname{tr}(\alpha_3\alpha_4), \quad \gamma_{12} = 3\operatorname{tr}(\alpha_2^2\alpha_3).
\end{gathered}
$$

Under our hypothesis, we know that all odd Ledger conditions are satisfied due to Theorem 3.1 and Remark 2.1. Thus, using Definition 2.1 and the fact that $\alpha_j(v) = \frac{1}{j!}C_v^{(j)}(0)$ for $j \geq 0$, we directly get $\gamma_2 = \operatorname{tr}\alpha_2 = \frac{1}{2!}\operatorname{tr}C_v^{(2)}(0) = 0$ and analogously, $\gamma_5 = 0 = \gamma_{10}$. Moreover, by (2), Proposition 2.2, (9) and Proposition 2.3, we also have

$$
\begin{aligned}
\gamma_6 = & \operatorname{tr}(2\alpha_2\alpha_3) = \tfrac{1}{6}\operatorname{tr}(C_v^{(2)}(0)C_v^{(3)}(0)) = \tfrac{1}{6}\operatorname{tr}(R_v \circ R_v') = 0, \\
\gamma_{11} = & 2\operatorname{tr}(\alpha_2\alpha_5) + 2\operatorname{tr}(\alpha_3\alpha_4) = \tfrac{1}{5!}\operatorname{tr}(C_v^{(2)}(0)C_v^{(5)}(0)) + \tfrac{1}{4!3}\operatorname{tr}(C_v^{(3)}(0)C_v^{(4)}(0)) \\
= & \tfrac{1}{54}\operatorname{tr}(R_v \circ R_v^{(3)}) + \left(\tfrac{1}{54} + \tfrac{1}{90}\right)\operatorname{tr}(R_v' \circ R_v^2) + \tfrac{1}{20}\operatorname{tr}(R_v' \circ R_v'') \\
= & \left(\tfrac{1}{54} + \tfrac{1}{90}\right)\operatorname{tr}(R_v' \circ R_v^2) + \left(\tfrac{1}{20} - \tfrac{3}{54}\right)\operatorname{tr}(R_v' \circ R_v'') \\
= & \tfrac{1}{540}\left(16\operatorname{tr}(R_v' \circ R_v^2) - 3\operatorname{tr}(R_v' \circ R_v'')\right) = 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
s_l = \operatorname{tr} C_v(t)^l = \; & n - 1 + t^2 \binom{l}{1}\gamma_1 + t^4 \left\{ \binom{l}{1}\gamma_3 + \binom{l}{2}\gamma_4 \right\} \\
& + t^6 \left\{ \binom{l}{1}\gamma_7 + \binom{l}{2}\gamma_8 + \binom{l}{3}\gamma_9 \right\} \\
& + t^7 \left\{ \binom{l}{3}\gamma_{12} \right\} + O(t^8)
\end{aligned} \tag{14}
$$

Now, denoting $\sigma_l = \sigma_l(v, t)$ and substituting (14) for $l = 1, ..., k$ in the recursive formula (11), we obtain

$$\sigma_k = \binom{n-1}{k} + t^2\binom{n-2}{k-1}\gamma_1 + t^4\left\{\binom{n-2}{k-1}\gamma_3 + \frac{1}{2}\binom{n-3}{k-2}\left(\gamma_1^2-\gamma_4\right)\right\}$$
$$+t^6\left\{\binom{n-2}{k-1}\gamma_7 + \binom{n-3}{k-2}\left(\gamma_1\gamma_3 - \frac{\gamma_8}{2}\right) + \binom{n-4}{k-3}\left(\frac{\gamma_1^3}{6} - \frac{\gamma_1\gamma_4}{2} + \frac{\gamma_9}{3}\right)\right\}$$
$$+t^7\left\{\frac{1}{3}\binom{n-4}{k-3}\gamma_{12}\right\} + O(v,t^8).$$

Moreover, it is easy to realize that $\gamma_i(v) = \gamma_i(-v)$, $i = 1,3,4,7,8,9$ and $\gamma_{12}(v) = -\gamma_{12}(-v)$ by Proposition 2.1 and (13). Thus, under the assumption that M is a k-D'Atri space for some $k \geq 3$ and setting $O(\pm v,t) = O(v,t) - O(-v,t)$, we obtain

$$0 = \sigma_k(v,t) - \sigma_k(-v,t) = t^7\left\{\frac{2}{3}\binom{n-4}{k-3}\gamma_{12}\right\} + O(\pm v, t^8).$$

This gives

$$\frac{2}{3}\binom{n-4}{k-3}\gamma_{12} + O(\pm v,t) = 0 \text{ for any small } t > 0,$$

which implies that $\gamma_{12} = 0$ for $n \geq 4$ and $k \geq 3$ taking into account that $\lim_{t\to 0} O(\pm v,t) = 0$. Therefore, proceeding as before,

$$0 = \gamma_{12} = 3\mathrm{tr}(\alpha_2^2\alpha_3) = \tfrac{1}{8}\mathrm{tr}(C_v^{(2)}(0)C_v^{(2)}(0)C_v^{(3)}(0)) = -\tfrac{1}{12}\mathrm{tr}(R_v^2 \circ R_v')$$

and we get the desired condition (12).

Finally, due to [14, Lemma 2.3] and (12), we have that

$$\frac{d}{dt}\left(\mathrm{tr}R^3_{\gamma_v'(t)}\right) = 3\mathrm{tr}\left(R^2_{\gamma_v'(t)} \circ R'_{\gamma_v'(t)}\right) = 0,$$

since $|\gamma_v'(t)| = 1$. Then, $\mathrm{tr}R^3_{\gamma_v'(t)} = \mathrm{tr}R_v^3$ along $\gamma_v(t)$ which means that $\mathrm{tr}R_v^3$ is invariant under the geodesic flow. □

4. k-D'Atri spaces of Iwasawa type

We recall that a solvable Lie algebra $\mathfrak{s}$ with inner product $\langle,\rangle$ is a metric Lie algebra of Iwasawa type, if it satisfies the conditions

(i) $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ where $\mathfrak{n} = [\mathfrak{s},\mathfrak{s}]$ and $\mathfrak{a}$, the orthogonal complement of $\mathfrak{n}$, is abelian.
(ii) The operators $\mathrm{ad}_H|_{\mathfrak{n}}$ are symmetric and non zero for all $H \in \mathfrak{a}$.
(iii) There exits $H_0 \in \mathfrak{a}$ such that $\mathrm{ad}_{H_0}|_{\mathfrak{n}}$ has all positive eigenvalues.

The simply connected Lie group S with Lie algebra $\mathfrak{s}$ and left invariant metric g induced by the inner product $\langle,\rangle$ will be called a space of Iwasawa type. The algebraic rank of S (equivalently $\mathfrak{s}$) is defined by $\dim\mathfrak{a}$.

In that follows we assume that $M = S$ and fix $m = e$, the identity of the group S; we identify $\mathfrak{s}$ with T_eS by $X = \widetilde{X}_e$, where $\widetilde{X}$ denotes the left invariant field on S associated to $X \in \mathfrak{s}$. The Levi Civita connection $\widetilde{\nabla}$ at

e, denoted by ∇, and the curvature tensor R associated to the metric can be computed by

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= \langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle \\ R(X,Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \end{aligned}$$

for any X, Y, Z in $\mathfrak{s}$. By a direct calculation using the above formula one obtains $\nabla_H = 0$ and hence $R_H = -\mathrm{ad}_H^2$, $R'_H = 0$ for all $H \in \mathfrak{a}$. In addition, if $\gamma_X(t)$, X a unit vector in $\mathfrak{s} = T_eS$, denotes the geodesic with $\gamma_X(0) = e$ and we express

$$\gamma'_X(t) = \left(dL_{\gamma_X(t)}\right)_e x(t) \text{ for all real } t,$$

with $x(t)$ the curve uniquely defined in the unit sphere of $\mathfrak{s}$, then

$$x(t) = H \quad (\gamma_X(t) = \exp tH) \text{ or } \lim_{t\to+\infty} x(t) = H \tag{15}$$

for some $H \in \mathfrak{a}$, by taking a sequence if necessary (see [13, Proposition 2.1]).

It is worth pointing out that irreducible homogeneous and simply connected D'Atri spaces of nonpositive curvature can be represented as Iwasawa type spaces and they are either symmetric spaces of higher rank or Damek-Ricci spaces in the case of rank 1, including the rank one symmetric spaces (see [15], [16]). Moreover, D'Atri spaces (without curvature restrictions) of Iwasawa type and algebraic rank one are Damek-Ricci spaces (see [13]). Now, we analyze $\mathfrak{C}$-spaces and the D'Atri property on Iwasawa type spaces.

Proposition 4.1. *If S is a space of Iwasawa type that is a $\mathfrak{C}$-space, then S is a nonpositively curved space.*

Proof. Let $X \in \mathfrak{s}$ be a unit vector. Since S is a $\mathfrak{C}$-space, the eigenvalues of $R_{\gamma'_X(t)}$ are constant along $\gamma_X(t)$ and the characteristic polynomial of $R_{\gamma'_X(t)}$ can be write as

$$\det\left(\lambda \mathrm{Id} - R_{\gamma'_X(t)}\right) = \det\left(\lambda \mathrm{Id} - R_X\right). \tag{16}$$

Moreover, using the previous notation

$$\begin{aligned} \det\left(\lambda \mathrm{Id} - R_{\gamma'_X(t)}\right) &= \det\left((\mathrm{d}L_{\gamma_X(t)})_e \circ (\lambda \mathrm{Id} - R_{x(t)}) \circ (\mathrm{d}L_{\gamma_X(t)})_e^{-1}\right) \\ &= \det\left(\lambda \mathrm{Id} - R_{x(t)}\right) \text{ for all } t \in \mathbf{R}. \end{aligned} \tag{17}$$

Then, taking lim as $t \to \infty$, it follows from (15) and (16) that

$$\det\left(\lambda \mathrm{Id} - R_X\right) = \det\left(\lambda \mathrm{Id} - R_H\right) = \det\left(\lambda \mathrm{Id} + \mathrm{ad}_H^2\right) \tag{18}$$

for some $H \in \mathfrak{a}$. Hence, (18) implies that the eigenvalues of R_X are exactly those of $-\mathrm{ad}_H^2$. Thus, the sectional curvature of S is nonpositive. $\square$

Corollary 4.2. *Let S be an irreducible space of Iwasawa type. Then, it is a symmetric space of noncompact type if and only if S is a D'Atri space and a $\mathfrak{C}$-space.*

Proof. It is well-known that symmetric spaces are an important subclass of D'Atri spaces and $\mathfrak{C}$-spaces.

On the other hand, by Proposition 4.1 S is an irreducible D'Atri space of nonpositive curvature. It follows from [15, Theorem 4.7] that either S is symmetric of noncompact type of higher rank, or S is a D'Atri (harmonic) space of algebraic rank one. In the last case, by applying [12, Corollary 2.2.] S is a symmetric space of rank one, since

$$\mathrm{tr} R_X^k = \mathrm{tr}\left(-\mathrm{ad}_{H_0}^2\right)^k \text{ for all } X \in \mathfrak{s},\ |X| = 1 \text{ and } k = 1, ..., n-1,$$

which means that S is k-stein for all $k = 1, ..., n-1$. □

In the next result we stablish a number of curvature conditions needed to determine whether a space of Iwasawa type is symmetric.

Theorem 4.3. *Let S be a space of Iwasawa type. Then,*

(i) *S is a symmetric space if and only if $\mathrm{tr}(32R_X^3 - 9R_X' \circ R_X')$ and $\mathrm{tr}(R_X^3)$ are invariant under the geodesic flow.*
(ii) *S is a symmetric space if and only if the three first odd Ledger conditions are satisfied and $\mathrm{tr}(R_X' \circ R_X^2) = 0$ for all unit vector $X \in \mathfrak{s}$.*

Proof. (i) If $\mathrm{tr}(32R_X^3 - 9R_X' \circ R_X')$ and $\mathrm{tr}(R_X^3)$ are invariant under the geodesic flow, then $\mathrm{tr}(R_X' \circ R_X')$ is also invariant under the geodesic flow. That is,

$$\mathrm{tr}(R'_{\gamma'_X(t)} \circ R'_{\gamma'_X(t)}) = \mathrm{tr}(R'_X \circ R'_X) \quad \text{ for all } \ t \in \mathbb{R}.$$

Finally, taking lim as $t \to \infty$, from (15) and the fact $R_H' = 0$ we have that

$$0 = \mathrm{tr}(R_H' \circ R_H') = \mathrm{tr}(R_X' \circ R_X'), \quad X \in \mathfrak{s},\ |X| = 1.$$

Hence, $R_X' = 0$ and consequently S is symmetric (see for example [19]).

(ii) If $L_3 = 0$, $L_5 = 0$, $L_7 = 0$ and $\mathrm{tr}(R_X' \circ R_X^2) = 0$ for all unit vector $X \in \mathfrak{s}$, we get $\mathrm{tr}(R_X' \circ R_X'') = 0$ by (7). Thus, $\mathrm{tr}(R_X' \circ R_X')$ is invariant under the geodesic flow and consequently, S is symmetric (see the proof of (i)).

□

Thus, from Remark 2.1 we characterize a special subclass of D'Atri spaces of Iwasawa type using only the three first odd Ledger's conditions.

Corollary 4.4. *Let S be a space of Iwasawa type. Then, S is a symmetric space if and only if S is D'Atri and $\mathrm{tr}(R_X' \circ R_X^2) = 0$ for all unit vector $X \in \mathfrak{s}$.*

Equivalently, if S is a D'Atri space of Iwasawa type, then S is symmetric if and only if $\mathrm{tr}(R_X^3)$ is invariant under the geodesic flow.

Remark 4.1. (i) *In particular, if the algebraic rank of S is 1, the property* $\mathrm{tr}(R_X^3)$ *being invariant under the geodesic flow is a necessary condition in the above corollary, since nonsymmetric Damek-Ricci spaces do not satisfy such property by* [12].

(ii) *The property* $\mathrm{tr}(R_X^3)$ *being invariant under the geodesic flow in* $\mathfrak{C}$-*spaces and the previous corollary give an alternative proof of Corollary 4.2.*

Finally, applying the main results of the preceding section and Theorem 4.3 we get some stronger results than the previously obtained in [14].

It is known that the property of S be a k-D'Atri space for all $k = 1, ..., n-1$ characterizes the symmetric spaces of noncompact type within the class of of Iwasawa type spaces. In particular, in the class of Damek-Ricci spaces the symmetric of noncompact type and rank 1 are characterized by the 3-D'Atri condition. (See [14, Theorem 3.2]). Now, we generalize this result in the class of Iwasawa type spaces, where the symmetric ones are those which satisfy the 3-D'Atri property.

Corollary 4.5. *Let S be a space of Iwasawa type. Then, S is symmetric if and only if S is* 3-*D'Atri.*

Proof. Applying Theorem 3.1, the equivalence between 1-D'Atri and 2-D'Atri properties and [14, Proposition 2.4], under the assumption that S is 3-D'Atri it follows that $\mathrm{tr}(R_X^k)$, $k = 1, 2, 3$ and $\mathrm{tr}(32R_X^3 - 9R'_X \circ R'_X)$ are invariant under the geodesic flow. The assertion is immediate by Theorem 4.3. □

Finally, we get an stronger consequence of Theorem 4.3 than the previously obtained in Corollary 4.5 using Theorem 3.2.

Corollary 4.6. *Let S be a space of Iwasawa type of dimension $n \geq 4$. Then, S is symmetric if and only if S is k-D'Atri for some $k \geq 3$.*

In particular, if S is Damek-Ricci then S is a k-D'Atri space for some $k \geq 3$ if and only if S is a rank one symmetric space of noncompact type.

References

[1] T. Arias-Marco, *Study of homogeneous D'Atri spaces of the Jacobi operator on g.o. spaces and the locally homogeneous connections on 2-dimensional manifolds with the help of* MATHEMATICA©. Dissertation, Universitat de València, Valencia, Spain (2007) ISBN: 978-84-370-6838-1, http://www.tdx.cat/TDX-0911108-110640.

[2] T. Arias-Marco, *The classification of 4-dimensional homogeneous D'Atri spaces revisited*, Differential Geom. Appl. **25** (2007), 29–34.

[3] T. Arias-Marco, *Some curvature conditions of Riemannian manifolds. Ledger's conditions and Jacobi osculating rank.* Lambert Academic Publishing, Saarbrücken, Germany, 2009.

[4] T. Arias-Marco, O. Kowalski, *The classification of 4-dimensional homogeneous D'Atri spaces*, Czechoslovak Math. J. **58(133)** (2008), 203–239.

[5] J. Berndt, F. Tricerri, L. Vanhecke, *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, Lecture Notes in Mathematics 1598, Springer-Verlag, Berlin/Heidelberg/New York, 1995.

[6] J. Berndt, L. Vanhecke, *Two natural generalizations of locally symmetric spaces*, Differential Geom. Appl. **2** (1992), 57–80.
[7] A.L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse der Mathematik und ihrer Grenzgebiete 93, Springer-Verlag, Berlin/New York, 1978.
[8] N. Bourbaki, *Algebra II (Chapters 4-7)*, Springer Verlag, Berlin 1990.
[9] B.-Y. Chen, L. Vanhecke, *Differential geometry of geodesic spheres*, J. Reine Angew. Math. **325** (1981), 28–67.
[10] J.E. D'Atri, H.K. Nickerson, *Divergence preserving geodesic symmetries*, J. Diff. Geom. **3** (1969), 467–476.
[11] M.J. Druetta, *$\mathfrak{C}$-spaces of Iwasawa type and Damek-Ricci spaces*, Contemporary Mathematics **288** (2001), 315–319.
[12] M.J. Druetta, *Carnot spaces and the k-stein condition*, Advances in Geometry **6** (2006), 439–465.
[13] M.J. Druetta, *D'Atri spaces of Iwasawa type*, Differential Geom. Appl. **27 (5)** (2009), 653–660.
[14] M.J. Druetta, *Geometry of D'Atri spaces of type k*, Ann. Glob. Anal. Geom. **38** (2010), 201–219.
[15] J. Heber, *Homogeneous spaces of nonpositive curvature and their geodesic flow*, International J. of Math. **6** (1995), 279–296.
[16] J. Heber, *On harmonic and asymtotically harmonic homogeneous spaces*, Geom. Funct. Anal. **16** (2006), 869–890.
[17] O. Kowalski, *Spaces with volume-preserving symmetries and related classes of Riemannian manifolds*, Rend. Sem. Mat. Univ. Politec. Torino, Fascicolo Speciale (1983), 131–158.
[18] O. Kowalski, F. Prüfer, L. Vanhecke, *D'Atri spaces*, Prog. Nonlinear Differential Equations Appl. **20** (1996), 241–284.
[19] R. Osserman, P. Sarnak, *A new curvature invariant and entropy of geodesic flows*, Invent. math. **77** (1984), 455–462.
[20] H. Pedersen, P. Tod, *The Ledger curvature conditions and D'Atri geometry*, Differential Geom. Appl. **11** (1999), 155–162.
[21] Z.I. Szabó, *Spectral theory for operator families on Riemannian manifolds*, Proc. Symp. Pure Math. **54**, 3 (1993), 615–665.

Departamento de Matemáticas, Universidad de Extremadura, 06006 Badajoz, Spain
E-mail address: ariasmarco@unex.es

CIEM - FaMAF, Universidad Nacional de Córdoba, 5000 Córdoba, Argentina
E-mail address: druetta@famaf.unc.edu.ar